

IDEAL OF A RING

Definition: A subring S of a ring R is said to be (i) A left ideal of R if $a \in S, r \in R \Rightarrow r.a \in S$; (ii) A right ideal of R if $a \in S, r \in R \Rightarrow a.r \in S$; (iii) a both sided ideal (or an ideal) of R if $a \in S, r \in R \Rightarrow r.a \in S$ and $a.r \in S$.

Let R be a ring. Then the improper subring R is an ideal of R . This ideal is called the improper ideal of R . All other ideals are called proper ideals of R .

The subring $\{0\}$ is also an ideal of R . This is called the trivial ideal or the null ideal of R .

Different types of ideal in a ring:

• Principal ideals in a ring: Let S be a non-empty subset of a ring R . The intersection of all ideals of R containing the subset S is an ideal of R and it is the smallest ideal of R containing the subset S . This is said to be the ideal generated by S .

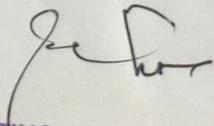
In particular, if S be a single element of R then the ideal generated by the element is called a principal ideal of R .

Thus corresponding to each element of a ring R there is a principal ideal of R . Definition: Let R be a ring and $a \in R$. The smallest ideal of R containing the element a is said to be a principal ideal of R . It is said to be the ideal generated by a and is denoted by $\langle a \rangle$.

Alternatively, an ideal U of a ring R is said to be a principal ideal of R if $U = \langle a \rangle$ for some a in R .

Examples.

1. Let R be a ring. The null ideal $\{0\}$ is the smallest ideal of R containing the element 0 . The null ideal $\{0\}$ is a principal ideal of R .
2. Let R be a ring with unity. Let 1 be the unity in R . Let U be the smallest ideal of R containing the element 1 . Since U is an ideal, $a \in R, 1 \in U \Rightarrow 1 \cdot a \in U$ i.e., $a \in U$. Thus $a \in R \Rightarrow a \in U$ and therefore $U = R$. Consequently, R is the principal ideal generated by 1 .
3. In the ring $2\mathbb{Z}$, the subring $\{0, \pm 4, \pm 8, \pm 12, \dots\}$ is an ideal. It is the smallest ideal containing the element 4 . Therefore it is the principal ideal $\langle 4 \rangle$.
4. Let us consider the ring $R = \mathbb{Z} \times \mathbb{Z}$, where $+$ are defined by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b)(c, d) = (ac, bd)$ for all $(a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z}$. The subring S of all ordered pairs $\{(a, 0) : a \in \mathbb{Z}\}$ is an ideal of R . We prove that it is the smallest ideal of R containing the element $(1, 0)$ of R .


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Let U be any ideal of R containing the element $(1,0)$. Let $(p, 0) \in S$. Then $(p, b) \in R$ for all $b \in Z$. Since U is an ideal of R , $(1, 0) \in U$, $(p, b) \in R \Rightarrow (1,0) \cdot (p, b) \in U$, i.e., $(p, 0) \in U$.

Therefore $S \subset U$. This shows that S is the smallest ideal of R containing the element $(1,0)$.

Hence S is a principal ideal of the ring R .

Principal ideal ring:

Definition: A ring is said to be a principal ideal ring if every ideal of the ring is a principal ideal.

Examples:

1. The ring Z is a principal ideal ring.

2. The ring Z , is a principal ideal ring.

● Prime ideals in a ring:

Definition: In a ring R , an ideal $P \neq R$ is said to be a prime ideal if for $a, b \in R$, $ab \in P$ implies either $a \in P$ or $b \in P$.

Examples:

1. The null ideal $\{0\}$ in the ring Z is a prime ideal, since for $a, b \in Z$, $ab \in \{0\}$ implies $ab = 0$ and this implies either $a = 0$ or $b = 0$ and this again implies either $a \in \{0\}$ or $b \in \{0\}$.

2. The ideal $2Z$ in the ring Z is a prime ideal. Let $ab \in 2Z$ for some $a, b \in Z$. $ab \in 2Z \Rightarrow ab = 2m$ for some integer m . This implies 2 is a divisor of ab and this again implies either 2 is a divisor of a or 2 is a divisor of b . 2 is a divisor of a implies $a \in 2Z$. 2 is a divisor of b implies $b \in 2Z$.

Thus $ab \in 2Z$ implies either $a \in 2Z$ or $b \in 2Z$. This proves that $2Z$ is a prime ideal in the ring Z .

3. The ideal $4Z$ in the ring $2Z$ is not a prime ideal, since $2 \cdot 2 \in 4Z$ but $2 \notin 4Z$.

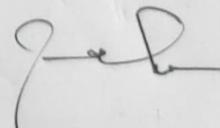
Prime ideals in the ring Z : The prime ideals in the ring Z are the ideals pZ , where p is either 0 or a prime.

Proof: The null ideal $\{0\}$ in the ring Z is a prime ideal, since for $a, b \in Z$ $ab \in \{0\}$ implies $ab = 0$ and this implies either $a = 0$ or $b = 0$ and this again implies either $a \in \{0\}$ or $b \in \{0\}$.

Let p be a prime. To prove that pZ is a prime ideal, let $ab \in pZ$ for some integers a, b . Then $ab = pm$ for some integer m . Therefore p is a divisor of ab . This implies either p is a divisor of a or p is a divisor of b . $p|a$ implies $a = pu$ for some integer u and therefore $a \in pZ$. $p|b$ implies $b = pv$ for some integer v and therefore $b \in pZ$. Thus $ab \in pZ$ implies either $a \in pZ$ or $b \in pZ$. This proves that pZ is a prime ideal in the ring Z . Conversely, let pZ be a non-null prime ideal in the ring Z . Therefore $p \neq 0$. Since pZ is a prime ideal, $pZ \neq Z$ and therefore $p \neq 1$. Let $p|ab$ for some integers a, b . Then $ab = pu$ for some integer u . Therefore $ab \in pZ$. Since pZ is a prime ideal, $ab \in pZ$ implies either $a \in pZ$ or $b \in pZ$. $a \in pZ \Rightarrow p|a$, $b \in pZ \Rightarrow p|b$. Thus $p|ab$ implies either $p|a$ or $p|b$. Therefore p is a prime. This completes the proof.

Maximal ideals in a ring:

Definition. Let R be a ring. An ideal $M \neq R$ is said to be a maximal ideal in R if for any ideal U of R satisfying $M \subset U \subset R$, either $U = M$ or $U = R$.



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In other words, a maximal ideal M of a ring R has the property that there exists no proper ideal of R strictly containing M .

Examples.

1. null ideal is a maximal ideal in a field F , since there is no proper ideal of F strictly containing the ideal $\{0\}$.
2. The null ideal is a maximal ideal in a simple ring.
3. In the ring $R = \mathbb{Z}$, the ideal $2\mathbb{Z}$ is a maximal ideal.

Let U be an ideal of the ring \mathbb{Z} such that $2\mathbb{Z} \subset U \subset \mathbb{Z}$. Since \mathbb{Z} is a principal ideal ring, U is a principal ideal of \mathbb{Z} . Let $U = (m)$ for some positive integer m . Then $U = m\mathbb{Z}$.

$2 \in 2\mathbb{Z} \subset m\mathbb{Z} \Rightarrow m|2$. This implies either $m = 1$ or $m = 2$. If $m = 1$ then $U = (1) = \mathbb{Z}$. If $m = 2$ then $U = (2) = 2\mathbb{Z}$. Therefore $2\mathbb{Z}$ is a maximal ideal in \mathbb{Z} .

This proves that the ideal $2\mathbb{Z}$ is a maximal ideal in the ring \mathbb{Z} .

4. In the ring $R = \mathbb{Z}$, the ideal $4\mathbb{Z}$ is not a maximal ideal, since the ideal $2\mathbb{Z}$ is a proper ideal of \mathbb{Z} strictly containing the ideal $4\mathbb{Z}$.
5. In the ring $R = \mathbb{Z} \times \mathbb{Z}$, let $S = \{(a, 0) : a \in \mathbb{Z}\}$, $T = \{(a, 2b) : a \in \mathbb{Z}, b \in \mathbb{Z}\}$.

Then S and T are ideals of R . T is a proper ideal of R strictly containing S . Therefore S is not a maximal ideal of R .

Maximal ideals in the ring \mathbb{Z} .

The ideal $p\mathbb{Z}$ in the ring \mathbb{Z} is maximal if and only if p is a prime.

Proof. Let p be a prime. To prove that the ideal $p\mathbb{Z}$ is maximal, let U be an ideal of the ring \mathbb{Z} properly containing the ideal $p\mathbb{Z}$. Then there is an element q in U such that $q \notin p\mathbb{Z}$. Clearly, q is not a multiple of p . Since p is a prime, $\gcd(p, q) = 1$. Therefore $pu + qv = 1$ for some integers u, v . $pu \in p\mathbb{Z} \subset U \Rightarrow pu \in U$; and $q \in U \Rightarrow qv \in U$.

Therefore $1 \in U$ and this implies $U = \mathbb{Z}$.

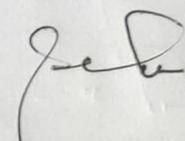
Thus no ideal of the ring \mathbb{Z} properly containing the ideal $p\mathbb{Z}$ is proper and this proves that the ideal $p\mathbb{Z}$ is maximal in \mathbb{Z} .

Conversely, let $p\mathbb{Z}$ be a maximal ideal in the ring \mathbb{Z} . Then $p\mathbb{Z} \neq \mathbb{Z}$ and therefore $p \neq 1$.

Let p be not a prime and let a be a divisor of p such that $1 < a < p$. a is a divisor of p implies $p \in a\mathbb{Z}$ and therefore $p\mathbb{Z} \subset a\mathbb{Z}$. $a < p$ implies $a \notin p\mathbb{Z}$. Since $a \in a\mathbb{Z}$, it follows that the ideal $a\mathbb{Z}$ properly contains the ideal $p\mathbb{Z}$. $a \neq 1$ implies the ideal $a\mathbb{Z}$ is a proper ideal of \mathbb{Z} .

Thus there is a proper ideal $a\mathbb{Z}$ properly containing the ideal $p\mathbb{Z}$ and so $p\mathbb{Z}$ is not maximal, a contradiction to the assumption. Therefore p is a prime.

This completes the proof.


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