PROJECT REPORT OF SEMESTER -3 OF THE DEPARMENT OF MATHEMATICS (HONS.) .

IDEAL OF A RING

Definition :A subring S of a ring R is said to be (i)A left ideal of R if a∈S, r∈ R ⇒ r.a ∈S; (ii)A right ideal of R if a∈S, r∈R⇒a.r∈S; (iii) a both sided ideal (or an ideal) of R if a∈S, r∈R⇒r.a∈S and a.r ∈S.

Let R be a ring. Then the improper subring R is an ideal of R. This ideal is called the improper ideal of R. All other ideals are called proper ideals of R.

The subring {0} is also an ideal of R. This is called the trivial ideal or the null ideal of R.

Different types of ideal in a ring:

•Principal ideals in a ring: Let S be a non-empty subset of a ring R. The intersection of all ideals of R containing the subset S is an ideal of R and it is the smallest ideal of R containing the subset S. This is said to be the ideal generated by S.

In particular, if S be a single element of R then the ideal generated by the element is called a principal ideal of R.

Thus corresponding to each element of a ring R there is a principal ideal of R. Definition: Let R be a ring and a∈R. The smallest ideal of R containing the element a is said to be a principal ideal of R. It is said to be the ideal generated by a and is denoted by (a).

Alternatively, an ideal U of a ring R is said to be a principal ideal of R if U= (a) for some a in R.

Examples.

1. Let R be a ring. The null ideal {0} is the smallest ideal of R containing the element 0. The null ideal {0} is a principal ideal of R.

2. Let R be a ring with unity. Let 1 be the unity in R. Let U be the smallest ideal of R containing the element 1. Since U is an ideal, a∈ R, 1∈U⇒1·a∈U i.e., a∈U.=Thus a∈R⇒ a∈U and therefore U=R. Consequently, R is the principal ideal generated by 1.

3. In the ring 2Z, the subring {0, ±4, ±8, ±12,...}is an ideal. It is the smallest ideal containing the element 4. Therefore it is the principal ideal <4>.

4. Let us consider the ring R = Zx Z, where + are defined by (a,b) + (c,d) = (a + c, b + d) and (a, b). (c,d)=(ab,cd) for all (a,b),(c,d)∈ZXZ. The subring S of all ordered pairs {(a,0): a∈Z} is an ideal of R. We prove that it is the smallest ideal of R containing the element (1,0) of R.

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Let U be any ideal of R containing the element (1,0). Let (p, 0)∈S. Then (p, b)∈R for all b ∈Z. Since U is an ideal of R, (1, 0) ∈U, (p, b) ∈ R⇒ (1,0).(p, b) U, i.e., (p, 0)∈U.

Therefore S⊂U. This shows that S is the smallest ideal of R con taining the element (1,0).

Hence S is a principal ideal of the ring R.

Principal ideal ring:

Definition: A ring is said to be a principal ideal ring if every ideal of the ring is a principal ideal.

Examples: **1.**The ring Z is a principal ideal ring.

2.The ring Z, is a principal ideal ring.

- Prime ideals in a ring:

Definition: In a ring R, an ideal P ≠ R is said to be a prime ideal f for a, b in R, ab∈P implies either a∈P or b∈P

Examples:

1.The null ideal {0} in the ring Z is a prime ideal, since for a,b ∈z ,ab ∈ {0} implies ab= 0 and this implies either a = 0 or b= 0 and this again implies either a ∈{0} or b∈{0}.

2. The ideal 2Z in the ring Z is a prime ideal. Let ab∈2Z for some a, b ∈ Z. ab ∈2Z ⇒ab = 2m for some integer m. This implies 2 is a divisor of ab and this again implies either 2 is a divisor of a or 2 is a divisor of b. 2 is a divisor of a implies a ∈2Z. 2 is a divisor of b implies b ∈2Z.
Thus ab ∈2Z implies either a ∈2Z or b∈ 2Z. This proves that 2Z is a prime ideal in the ring Z.

3. The ideal 4Z in the ring 2Z is not a prime ideal, since 2.2 ∈ 4Z but 2 ∉4Z.

Prime ideals in the ring Z: The prime ideals in the ring Z are the ideals pZ, where p is either 0 or a prime.

Proof: The null ideal {0} in the ring Z is a prime ideal, since for a, b ∈ Z ab ∈ {0} implies ab = 0 and this implies either a = 0 or b = 0 and this again implies either a ∈ ({0} or b ∈ {0}.

Let p be a prime. To prove that pZ is a prime ideal, let ab ∈ pZ form some integers a, b. Then ab = pm for some integer m. Therefore p is a divisor of ab. This implies either p is a divisor of a or p is a divisor of b. P|a implies a = pu for some integer u and therefore a ∈ pZ. P|b implies b=pu for some integer u and therefore b ∈ pZ. Thus ab ∈ pZ implies either a ∈ pz or b ∈ pZ. This proves that pZ is a prime ideal in the ring Z. Conversely, let pZ be a non-null prime ideal in the ring Z. Therefore p ≠ 0. Since pZ is a prime ideal, pZ ≠ Z and therefore p ≠ 1. Let p|ab for some integers a, b. Then ab=pu for some integer Therefore ab ∈ pZ. Since pZ is a prime ideal, ab ∈ pZ implies either a ∈pZ or b ∈ pZ. A ∈ pZ⇒p|a, b ∈ pZ ⇒p|b. Thus p|ab implies either p|a or p|b. Therefore p is a prime. This completes the proof.

Maximal ideals in a ring:

Definition. Let R be a ring. An ideal M≠R is said to be a maximal ideal in R if for any ideal U of R satisfying M⊂U⊂R, either U=M or U= R.

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In other words, a maximal ideal M of a ring R has the property that there exists no proper ideal of R strictly containing M.

Examples.

1. null ideal is a maximal ideal in a field F, since there is no proper ideal of F strictly containing the ideal $\{0\}$.

2. The null ideal is a maximal ideal in a simple ring.

3. In the ring R= Z, the ideal $2Z$ is a maximal ideal.

Let U be an ideal of the ring Z such that $2Z \subset U \subset R$. Since Z is a principal ideal ring, U is a principal ideal of Z. Le U = (m) for some positive integer m. Then U = mZ.

$2 \in 2Z \subset mZ \Rightarrow m|2$. This implies either m = 1 or m = 2. If m = 1 then U = (1) = R. If m=2 then U = (2) = 2Z. Therefore $2Z \subset U \subset R$ imples either U = R or U = 2Z.

This proves that the ideal 2Z is a maximal ideal in the ring Z.

4. In the ring R = Z, the ideal 4Z is a not a maximal ideal, since the Ideal 2Z is a proper ideal of R strictly containing the ideal 4Z.

5. In the ring R = Z x Z, let $S = \{(a,0): a \in Z\}$, $T = \{(a, 2b): \in Z, b \in Z\}$.

Then S and T are ideals of R. T is a proper ideal of R strictly containing S. Therefore S is not a maximal ideal of R.

Maximal ideals in the ring Z.

The ideal pZ in the ring Z is maximal if and only if p is a prime.

Proof. Let p be a prime. To prove that the ideal pZ is maximal, let U be an ideal of the ring Z properly containing the ideal pZ. Then there is an element q in U such that $q \notin pZ$. Clearly, q is not a multiple of p. Since p is a prime, gcd(p, q) = 1. Therefore pu + qv = 1 for some integers u, v. $pu \in pZ \subset U \Rightarrow pu \in U$; and $q \in U \Rightarrow qv \in U$.

Therefore $1 \in U$ and this implies U = Z.

Thus no ideal of the ring Z properly containing the ideal pZ is proper and this proves that the ideal pZ is maximal in Z.

Conversely, let pZ be a maximal ideal in the ring Z. Then $pZ \neq Z$ and therefore $p \neq 1$.

Let p be not a prime and let a be a divisor of p such that $1 < a < p$. a is a divisor of p implies $p \in aZ$ and therefore $pZ \subset aZ$. $A < p$ implies $a \notin pZ$. Since $a \in aZ$, it follows that the ideal aZ properly contains the Ideal pz. $A \neq 1$ implies the ideal aZ is a proper ideal of Z.

Thus there is a proper ideal aZ properly containing the ideal pz and so pZ is not maximal, a contradiction to the assumption. Thefore p is a prime.

This completes the proof.